ON THE EXPLOSION OF LINEARLY DISTRIBUTED CHARGE WITH CURVILINEAR SHAPE ON THE GROUND SURFACE

L. M. Kotlyar

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We use the momentum version proposed by M. A. Lavrent'yev [1, 2] to treat the two-dimensional problem of the explosion of a linearly distributed charge of curvilinear shape on the ground surface. The problem of the explosion of a straight charge was solved for the first time in [2] in this version. The ground is assumed to be an ideal incompressible liquid at velocities exceeding some critical velocity which remains constant along the crater; beyond this boundary, the medium is fixed. The potential of the velocity is assumed to be constant on the charge and vanishing on the ground surface.

1. Let us consider the two-dimensional stationary potential flow of an ideal weightless liquid in a portion of the plane $z=x+iy$; the flow is limited by the curvilinear section D'AD, the straight boundaries D'C' and DC on which the potential of the velocity is constant, and the flow line C'BC on which the absolute value of the velocity is constant and equal to V_0. The x axis runs vertically downward and is the symmetry axis of the flow; the y axis is parallel to the horizontal surface (Fig. 1a).

In view of the symmetry, we will consider only the right half of the flow pattern.

In order to solve the problem, we use the auxiliary complex variable $u=\xi+i\cdot\eta$ which varies in a region G (rectangle with the sides $\pi/4$ and $\pi\cdot\tau/4$; ($\tau=i|\tau|$); we will determine the function $z(u)$ which provides for the conformal mapping of the region G upon the flow region; we will obtain the correspondence of the points as shown in Fig. 1b.

According to [2], we introduce the boundary conditions for the complex potential $W(u)=\varphi+i\psi$:

$$\begin{aligned}&\operatorname{Re} W=\varphi=\varphi_0,\ u=\frac{\pi}{4}+i\eta,\\&\operatorname{Re} W=\varphi=0,\quad u=\xi,\\&\operatorname{Im} W=\psi=\text{const},\ u=i\eta,\ u=\xi+\frac{\pi\cdot\tau}{4}.\end{aligned}\tag{1.1}$$

It follows from Eq. (1.1) that the function dW/du is purely imaginary on BCD and purely real on BAD. This means that the function can be continued over the entire plane in accordance with the symmetry principle. In the region G(dW/du) has a first-order zero at the point B and a first-order pole at the point D (vortex). We obtain from the theory of elliptic functions [3]

$$-iNF(u)=\frac{dW}{du}=-iN\frac{\vartheta_1\left(u-\frac{\pi\tau}{4}\right)\vartheta_1\left(u+\frac{\pi\tau}{4}\right)\vartheta_2\left(u-\frac{\pi\tau}{4}\right)\vartheta_2\left(u+\frac{\pi\tau}{4}\right)}{\vartheta_1\left(u-\frac{\pi}{4}\right)\vartheta_1\left(u+\frac{\pi}{4}\right)\vartheta_4\left(u-\frac{\pi}{4}\right)\vartheta_4\left(u+\frac{\pi}{4}\right)},\tag{1.2}$$

where N denotes a real positive constant, and $\vartheta_k(u)$ denotes the theta functions for the periods π and $\pi\tau$ [3].

After determining from Eqs. (1.2) W(u) at the point D, we express the constant N through φ_0

$$N=\frac{\varphi_0}{\pi}M,\quad M=2\left[\frac{\vartheta_2\vartheta_3\vartheta_4}{\left|\vartheta_1\left(\frac{\pi}{4}+\frac{\pi\tau}{4}\right)\vartheta_2\left(\frac{\pi}{4}+\frac{\pi\tau}{4}\right)\right|}\right]^2,\tag{1.3}$$

where $\vartheta_k=\vartheta_k(0)$.

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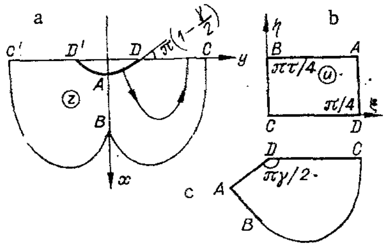

Fig. 1

Let us consider the Zhukovskii function $X(u)=\ln(1/V)(dW/dz)=\ln[(V/V_0)-i\Theta]=r-i\Theta$, where V denotes the absolute value of the velocity, and Θ denotes the angle by which the velocity vector is inclined relative to the x axis.

On the straight boundary sections, the function X(u) must satisfy the following conditions:

$$\operatorname{Im} X(u) = -\Theta = -\pi, \quad u=\xi, \quad \operatorname{Im} X(u) = -\Theta = 0, \quad u=\xi+\frac{\pi\tau}{4}. \tag{1.4}$$

On the free boundary $V=V_0$ and

$$\operatorname{Re} X(u) = r = 0, \quad u = i\eta. \tag{1.5}$$

Assume that the angle $\beta(s)$ enclosed by the tangent with the abscissa axis is given on the curvilinear arc AD, where s denotes the length of the arc reckoned from the point A and referred to the total length l of the arc AD. The dimensionless curvature of the arc is $\varkappa(\beta)=d\beta/ds$ and, since $\beta=\Theta+\pi/2$, we have

$$\varkappa(\Theta) = \frac{d\Theta}{ds}. \tag{1.6}$$

We obtain with Eqs. (1.2), (1.3), and (1.6) the boundary condition for X(u) at $u=(\pi/4)+i\eta$

$$\frac{d\Theta}{d\eta} = \delta\,\frac{M}{\pi}\,\varkappa(\Theta)\left|F\left(\frac{\pi}{4}+i\eta\right)\right|e^{-r(\eta)}, \tag{1.7}$$

where $\delta=\varphi_0/V_0 l$ denotes a dimensionless parameter.

We try to determine the function X(u) in the form

$$X(u) = X_*(u) - f(u), \tag{1.8}$$

where $f(u)=\mu+i\varepsilon$ is a function analytic on G and continuous in $\bar{G}$; $X_*(u)=r_*-i\Theta_*$ satisfies the following boundary conditions:

$$\begin{aligned}
&\operatorname{Im} X_*(u) = -\Theta_* = -\pi, \quad u=\xi,\\
&\operatorname{Im} X_*(u) = -\Theta_* = -\pi\left(1-\frac{\gamma}{2}\right), \; u=\xi+\frac{\pi\tau}{4}, \quad u=\frac{\pi}{4}+i\eta,\\
&\operatorname{Re} X_*(u) = r_* = 0, \quad u=i\eta,
\end{aligned} \tag{1.9}$$

where $\pi[1-(\gamma/2)]$ denotes the angle enclosed by the tangent to the line AD at the point D with the y axis (Fig. 1a).

It follows from Eq. (1.9) that the function $X_*(u)$ is the Zhukovskii function for the two-dimensional flow of an ideal weightless liquid according to the scheme of Fig. 1c; this scheme is obtained by replacing the arc AD by a straight section and by rotating the flow line AB by the angle $\pi[1-(\gamma/2)]$.

The derivative of the function $X_*(u)$ is purely real on CD and BA and purely imaginary on BC and AD. By representing the derivative as a linear combination of logarithmic derivatives of the theta functions [3] and by integrating, we obtain

$$X_*(u) = i\pi(\gamma-1) + \gamma\ln\frac{\vartheta_1\left(u+\frac{\pi}{4}\right)\vartheta_4\left(u+\frac{\pi}{4}\right)}{\vartheta_1\left(u-\frac{\pi}{4}\right)\vartheta_4\left(u-\frac{\pi}{4}\right)}. \tag{1.10}$$

Comparing the boundary conditions (1.4), (1.5), and (1.7) for X(u) with Eq. (1.9) for $X_*(u)$, we obtain the boundary conditions for the unknown function f(u):

$$\varepsilon_\eta{}^1 = \delta\,\frac{M}{\pi}\,\varkappa(\Theta)\,\nu(\eta)\,e^{\mu(\eta)}, \quad u=\frac{\pi}{4}+i\eta, \tag{1.11}$$

$$\begin{aligned}
&\operatorname{Re} f(u) = \mu = 0, \quad u=i\eta,\\
&\operatorname{Im} f(u) = \varepsilon = 0, \quad u=\xi,\\
&\operatorname{Im} f(u) = \varepsilon = -\pi\left(1-\frac{\gamma}{2}\right), \quad u=\frac{\pi\tau}{4}+\xi,
\end{aligned} \tag{1.12}$$

where $\nu(\eta) = |F[(\pi/4)+i\eta]|\exp(-r_*(\eta))$.

The Woods formula [4] can be used to introduce an operator defining on AD the real part of a function which is analytic on G; this operator is given by the derivative of the imaginary part on AB and satisfies the boundary conditions (1.11). It is possible to show that the boundary-value problem (1.11), (1.12) can be uniquely solved for sufficiently small values of the angle of the arc.

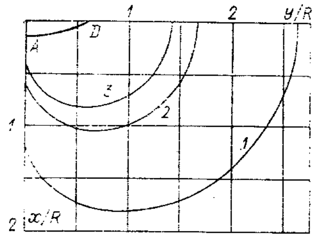

Fig. 2

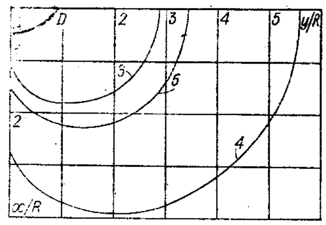

Fig. 3

2. We map the region G upon the half-ring $\rho \leq |\zeta| \leq 1$ with the aid of the function

$$\zeta = \exp\left(\frac{4u-\pi}{|\tau|}\right), \quad \rho = e^{-\frac{\pi}{|\tau|}} \tag{2.1}$$

and consider the function

$$p(\zeta) = f(u(\zeta)) + \ln \zeta\left(1 - \frac{\gamma}{2}\right). \tag{2.2}$$

According to Eq. (1.9), this function must satisfy the boundary conditions

$$\operatorname{Im} p(\zeta) = 0, \quad \operatorname{Im} \zeta = 0. \tag{2.3}$$

This means that $P(\zeta)$ can be continued over the entire ring according to the symmetry principle and can be represented as a Laurent series on the ring:

$$p(\zeta) = \sum_{n=-\infty}^{\infty} c_n \zeta^n, \tag{2.4}$$

where c_n denotes real coefficients.

We obtain with Eqs. (2.1) and (2.2) from Eq. (2.4)

$$f(u) = -\frac{(4u-\pi)\left(1-\frac{\gamma}{2}\right)}{|\tau|} + \sum_{n=-\infty}^{\infty} c_n e^{-\frac{4u-\pi}{|\tau|} \cdot n}. \tag{2.5}$$

We obtain from the first boundary condition (1.8):

$$\operatorname{Re} f(u)_{u=i\eta} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{-\frac{\pi n}{|\tau|}} + c_{-n} e^{\frac{\pi n}{|\tau|}}\right)\cos\frac{n4\eta}{|\tau|} + \frac{\pi\left(1-\frac{\gamma}{2}\right)}{|\tau|} = 0, \tag{2.6}$$

$$c_0 = -\frac{\pi}{|\tau|}\left(1 - \frac{\gamma}{2}\right), \quad c_{-n} = -c_n \rho^{2n}, \quad \rho = e^{-\frac{\pi}{|\tau|}}.$$

By substituting the expressions for the real and imaginary parts of f(u) from Eq. (2.5) into Eq. (1.12), we obtain with Eq. (2.6)

$$\sum_{n=1}^{\infty} c_n (1-\rho^{2n}) \frac{4n}{|\tau|} \cos\frac{4n}{|\tau|}\eta - 4\frac{\left(1-\frac{\gamma}{2}\right)}{|\tau|} = \delta \frac{M}{\pi} e^{-\frac{\pi}{|\tau|}\left(1-\frac{\gamma}{2}\right)} \varkappa(\Theta) Q(\eta), \tag{2.7}$$

$$Q(\eta) = \frac{\left|\vartheta_2\left(\frac{\pi}{4} + i\eta - \frac{\pi\tau}{4}\right)\vartheta_2\left(\frac{\pi}{4} + i\eta + \frac{\pi\tau}{4}\right)\right|^2}{|\vartheta_2(i\eta)\,\vartheta_3(i\eta)|^{1+\gamma}\,|\vartheta_1(i\eta)\,\vartheta_4(i\eta)|^{1-\gamma}} \exp\left[\sum_{n=1}^{\infty} c_n (1-\rho^{2n}) \cos\frac{4n}{|\tau|}\eta\right].$$

Integrating Eq. (2.7) over η from 0 to $\pi|\tau|/4$, we obtain a condition which the coefficients c_n must satisfy:

$$-\pi\left(1-\frac{\gamma}{2}\right) = \delta\frac{M}{\pi} e^{-\frac{\pi}{|\tau|}\left(1-\frac{\gamma}{2}\right)} I_0, \tag{2.8}$$

$$I_0 = \int_0^{\frac{\pi|\tau|}{4}} \varkappa(\Theta) Q(\eta)\, d\eta.$$

By multiplying Eq. (2.8) with $\cos(4n/|\tau|)\eta$ and integrating this equation within the above limits, we obtain

$$c_n = -\frac{2\left(1-\frac{\gamma}{2}\right)}{n(1+\rho^{2n})} \cdot \frac{I_n}{I_0}, \quad I_n = \int_0^{\frac{\pi|\tau|}{4}} \varkappa(\Theta) Q(\eta) \cos\frac{4n}{|\tau|}\eta\, d\eta. \tag{2.9}$$

Equation (2.9) is used to determine by iteration the coefficients c_n at given $|\tau|$ and γ.

After determining the function $f(u)$, all geometrical and hydrodynamic characteristics of the flow can be easily determined from Eqs. (1.2), (1.8), and (1.10).

3. We determined as an example the form of the crater resulting from the explosion of a charge with circular shape. In this case for $\delta = (\varphi_0/V_0)R$ (with R denoting the radius of the arc AB), we have $\varkappa(\Theta) = 1$. The parameter δ is obtained from Eqs. (1.3) and (2.8)

$$\delta = \frac{\pi}{M} a, \quad \left(a = \frac{\pi\left(1 - \frac{\gamma}{2}\right) \rho^{-\left(1 - \frac{\gamma}{2}\right)}}{|I_0|} \right). \tag{3.1}$$

The coordinates of the boundary BC are obtained from Eqs. (1.2), (1.8), and (1.10):

$$\begin{Bmatrix} x/R \\ y/R \end{Bmatrix} = a I_{x,y}, \qquad I_{x,y} = \int_0^{\eta} \frac{\left| \vartheta_1\left(i\eta - \frac{\pi\tau}{4}\right) \vartheta_1\left(i\eta + \frac{\pi\tau}{4}\right) \vartheta_2\left(i\eta - \frac{\pi\tau}{4}\right) \vartheta_2\left(i\eta + \frac{\pi\tau}{4}\right) \right|}{\left| \vartheta_1\left(\frac{\pi}{4} + i\eta\right) \cdot \vartheta_4\left(\frac{\pi}{4} + i\eta\right) \right|^2} \times$$

$$\times \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left[-4\eta\left(1 - \frac{\gamma}{2}\right) + 2\sum_{n=1}^{\infty} c_n \rho^n \sin \frac{4n}{|\tau|}\eta - 2\gamma \arg \vartheta_1\left(i\eta + \frac{\pi}{4}\right) - 2\gamma \arg \vartheta_4\left(i\eta + \frac{\pi}{4}\right) \right] d\eta.$$

The shapes of the ejection craters are shown in Fig. 2 ($\gamma = 5/3$) and Fig. 3 ($\gamma = 4/3$). Curves 1-6 correspond to the $\delta = 5.05$; 2.52; 1.93; 10.81; 5.28; 3.96, respectively.

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